

# Generalising a finite version of Euler's partition identity

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## Abstract

Recently, George Andrews has given a Glaisher style proof of a finite version of Euler's partition identity. We generalise this result by giving a finite version of Glaisher's partition identity. Both the generating function and bijective proofs are presented.

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## 1 Introduction

The so-called Euler's theorem has been widely studied. In the language of integer partitions, the theorem implies that the number of partitions of  $n$  into odd parts is equal to the number of partitions of  $n$  into distinct parts. This identity, called Euler's partition identity, has been refined (see [2]).

J. W. L. Glaisher gave a bijective proof of the identity (see [1]). Furthermore, its finite version was given together with bijective proofs (see [4], [3]). We recall this version below.

**Theorem 1.1** (Euler's theorem-finite version). *The number of partitions of  $n$  into odd parts each  $\leq 2N$  equals the number of partitions of  $n$  into parts each  $\leq 2N$  in which the parts  $\leq N$  are distinct.*

For example, if  $n = 10$ ,  $N = 3$ , then the seven partitions of  $n$  into odd parts each  $\leq 6$  are

$$5 + 5, 5 + 3 + 1 + 1, 5 + 1 + 1 + 1 + 1 + 1, 3 + 3 + 3 + 1,$$

$3+3+1+1+1+1, 3+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1+1$ .

And the seven partitions with parts  $\leq 6$  such that parts  $\leq 3$  are distinct are

$$6+1, 6+3+1, 5+5, 5+4+1, 5+3+2, 4+4+2, 4+3+2+1.$$

However, the bijections for Theorem 1.1 given in [3] are complicated, and motivated by their complexity, George Andrews gave a much simpler proof that is Glaisher style (see [1]).

It is clear that Euler's partition identity is a specific case of Glaisher's partition identity (see Theorem 1.2) when  $s = 2$ .

**Theorem 1.2** (Glaisher's identity, [5]). *The number of partitions of  $n$  into parts not divisible by  $s$  is equal to the number of partitions of  $n$  into parts not repeated more than  $s - 1$  times.*

We are then naturally led to ask as to whether a finite version of Glaisher's partition identity that generalises Theorem 1.1 is possible. If so, can we find a bijective proof thereof reminiscent of George Andrews Glaisher style proof?

The goal of this paper is to fully address the questions above. Our main result is as follows:

**Theorem 1.3.** *Let  $s$  be a positive integer. The number of partitions of  $n$  into parts not divisible by  $s$  each  $\leq sN$  equals the number of partitions of  $n$  into parts each  $\leq sN$  in which the parts  $\leq N$  occur at most  $s - 1$  times.*

Observe that Theorem 1.1 is obtained by setting  $s = 2$ . In the subsequent section, we give a generating function proof of the result, and in the section thereafter, a bijective proof that is Glaisher style.

## 2 First Proof of Theorem 1.3

Let  $\mathcal{O}_{s,N}(n)$  denote the number of partitions of  $n$  in which each part is not divisible by  $s$  and  $\leq sN$ , and  $\mathcal{D}_{s,N}(n)$  denote the number of partitions of  $n$  in which each part is  $\leq sN$  and all parts  $\leq N$  occur at most  $s - 1$  times. Thus

$$\sum_{n=0}^{\infty} \mathcal{O}_{s,N}(n)q^n = \prod_{n=1}^N \frac{1}{(1 - q^{sn-1})(1 - q^{sn-2}) \dots (1 - q^{sn-s+1})}$$

and

$$\sum_{n=0}^{\infty} \mathcal{D}_{s,N}(n)q^n = \frac{\prod_{n=1}^N (1 + q^n + q^{2n} + \dots + q^{(s-1)n})}{\prod_{n=1}^{(s-1)N} (1 - q^{n+N})}.$$

Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{D}_{s,N}(n)q^n &= \frac{\prod_{n=1}^N (1 + q^n + q^{2n} + \dots + q^{(s-1)n})}{\prod_{n=1}^{(s-1)N} (1 - q^{n+N})} \\
&= \frac{\prod_{n=1}^N (1 - q^n)(1 + q^n + q^{2n} + \dots + q^{(s-1)n})}{\prod_{n=1}^N (1 - q^n) \prod_{n=1}^{(s-1)N} (1 - q^{n+N})} \\
&= \frac{\prod_{n=1}^N (1 - q^{sn})}{\prod_{n=1}^{sN} (1 - q^n)} \\
&= \prod_{n=1}^N \frac{1}{(1 - q^{sn-1})(1 - q^{sn-2}) \dots (1 - q^{sn-s+1})} \\
&= \sum_{n=0}^{\infty} \mathcal{O}_{s,N}(n)q^n.
\end{aligned}$$

### 3 Second Proof of Theorem 1.3

We give a simple Glaisher style extension of the bijection given by George Andrews [1].

Consider a partition  $\lambda$  enumerated by  $\mathcal{O}_{s,N}(n)$ . Each part is of the form  $sm - t$  for some  $t = 1, 2, \dots, s-1$ . We can rewrite the partition as

$$\sum_{t=1}^{s-1} \sum_{i=1}^{r_t} f_{i,t}(sm_i - t)$$

where  $f_{i,t}$  is the multiplicity of the part  $sm_i - t$ ,  $r_t$  is the number of parts that are  $\equiv -t \pmod{s}$ .

Note that there exists a unique  $\alpha_{i,t}$  such that  $N < (sm_i - t)s^{\alpha_{i,t}} \leq sN$ . Rather than taking a complete  $s$ -ary expansion of  $f_{i,t}$ , we instead do the following: we find the aforementioned  $\alpha_{i,t}$  and use division algorithm to compute  $\beta_{i,t}$  and  $e_{i,t}$  from the equation

$$f_{i,t} = \beta_{i,t}s^{\alpha_{i,t}} + e_{i,t} \text{ where } 0 \leq e_{i,t} \leq s^{\alpha_{i,t}} - 1.$$

Then write the  $s$ -ary expansion of  $e_{i,t}$ , i.e.,

$$e_{i,t} = \sum_{j=0}^{b_i} a_{j,t}s^j \text{ where } 0 \leq a_{j,t} \leq s-1.$$

So  $f_{i,t} = \sum_{j=0}^{b_i} a_{j,t} s^j + \beta_{i,t} s^{\alpha_{i,t}}$  and thus

$$\begin{aligned}
\sum_{t=1}^{s-1} \sum_{i=1}^{r_t} f_{i,t}(sm_i - t) &= \sum_{t=1}^{s-1} \sum_{i=1}^{r_t} \left( \sum_{j=0}^{b_i} a_{j,t} s^j + \beta_{i,t} s^{\alpha_{i,t}} \right) (sm_i - t) \\
&= \sum_{t=1}^{s-1} \sum_{i=1}^{r_t} \sum_{j=0}^{b_i} a_{j,t} (sm_i - t) s^j + \sum_{t=1}^{s-1} \sum_{i=1}^{r_t} \beta_{i,t} (sm_i - t) s^{\alpha_{i,t}} \\
&= \sum_{t=1}^{s-1} \sum_{i=1}^{r_t} (a_{0,t}(sm_i - t) + a_{1,t}(sm_i - t)s + \dots \\
&\quad \dots + a_{b_i,t}(sm_i - t)s^{b_i}) + \sum_{t=1}^{s-1} \sum_{i=1}^{r_t} \beta_{i,t}(sm_i - t)s^{\alpha_{i,t}}
\end{aligned}$$

which is the image of  $\lambda$  with parts  $(sm_i - t)s^j \leq N$  that have multiplicity  $a_{j,t} \leq s - 1$ , and parts  $(sm_i - t)s^{\alpha_{i,t}} \in [N + 1, sN]$  that have multiplicity  $0 \leq \beta_{i,t} < \infty$ . This image is a partition enumerated by  $\mathcal{D}_{s,N}(n)$ .

The inverse of the bijection is not difficult to construct.

We demonstrate the bijection using  $n = 177, s = 3, N = 4$  and

$$\lambda = (11^6, 7^5, 5^7, 4^5, 2^2, 1^{17}).$$

Note that  $11.3^0 = 11.1 \in [5, 12]$ ,  $7.3^0 = 7.1 \in [5, 12]$ ,  $5.3^0 = 5.1 \in [5, 12]$ ,  $4.3^1 = 4.3 \in [5, 12]$ ,  $2.3^1 = 2.3 \in [5, 12]$ , and  $1.3^2 = 1.9 \in [5, 12]$ .

$11^6$  is mapped to  $11(6.1 + 0) = 6.11$ , which is interpreted as  $11^6$ . Similarly,  $7^5$  to  $7^5$ , and  $5^7$  to  $5^7$ .

$4^5$  goes to  $4(1.3 + 2)$ , using division algorithm  $5 = 1.3 + 2$ , and now taking the 3-ary expansion of 2;  $4(1.3 + 2) = 4(1.3 + 2.3^0) = 1.12 + 2.4$ , which is  $(12, 4^2)$ . Continuing in this manner, we have the mapping  $2^2 \mapsto 2^2$  and  $1^{17} \mapsto (9, 3^2, 1^2)$ . Thus

$$\lambda \mapsto (12, 11^6, 9, 7^5, 5^7, 4^2, 3^2, 1^2).$$

## References

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